Frictional Pressure Drop in Two-Phase Flow: B. An Approach Through Similarity Analysis

A. E. DUKLER, MOYE WICKS, III, and R. G. CLEVELAND

University of Houston, Houston, Texas

In searching for a method to correlate data for frictional pressure drop in two-phase flow it is of interest to examine the approaches which have been used in the past on other complex problems involving transport processes. When one eliminates the class of problems which have been solved by the use of potential theory or those which have used the assumptions of laminar flow in situations of relatively simple geometry, there remains those problems for which it has been necessary to develop design correlations without the availability of rigorous methods. This is not an unusual assignment for the chemical engineer. Typically his analysis of a complex problem produces very imperfect correlations at first based on limited data. These correlations improve as the problem receives further study and more sophisticated tools are used for its analysis. Between the first approaches to correlation and the refined attacks which usually appear many years later there appear to be a number of distinctly different approaches which have been employed. These can perhaps be classified as follows:

1. Empirical correlation.
2. Correlations with dimensional analysis used.
3. Correlations with similarity analysis and model theory used.
4. Mathematical analysis of a simplified physical model and development of equations relating the variables.
5. Solutions to the energy, momentum, and conservation equations using empirical expressions for the turbulent transport terms, approximations to the boundary conditions, and assumptions as the relative magnitude of the various terms in the equations. Usually the resulting relationship between variables is obtained by numerical solution of the complex equations.

On any problem which has resisted definitive solution over the years examples can usually be found of the application of each of these approaches.

Attempts to develop correlations between frictional pressure drop and the controlling variables during gas-liquid flow in conduits have utilized only three of these methods. Many empirical correlations have appeared. Most of these can be used beyond the range of the data from which they were constructed with poor reliability, as has been shown in the first paper (*1*). Correlations of this type for horizontal flow appear in references *2, 3, 4, 5, 6, 7*. Empirical correlations for vertical conduits are given in reference *8, 9, 10, 11*. The correlations which have demonstrated the greatest success have been those based on a simplified physical model of the complex two-phase system. The homogeneous model has received considerable attention because it permits the two phases to be treated as an equivalent single phase having mixture properties (*12, 13, 14, 15*). The annular flow model has produced a number of correlations (*16, 17, 18, 19*). Recently an approach through the solution of the equations of motion was presented (*20*). Using the Prandtl mixing length theory, and the equations for potential flow to develop an equation for the density distribution, Levy solved the equations of motion by numerical means. The resulting relationships between shear stress and the operational variables show agreement only with very restricted data.

No significant attempts have been reported at correlating the two-phase frictional pressure drop by the use of dimensional analysis or similarity analysis. The use of dimensionless groups for empirical correlations techniques is widespread; however a correlation based primarily on dimensional analysis has not appeared. For a process with the large number of variables involved in two-phase flow it is doubtful that the use of dimensional analysis *alone* can provide a fruitful approach. It is readily demonstrated that four dimensionless groups are involved for each phase. Thus a total of eight dimensionless groups must be considered, and in each of these the phase velocity is ununknown. Therefore the use of experimental data to provide the interrelating constants becomes a near impossible task.

No approach to correlation using the principles of similarity in a formal manner has appeared in the literature. It is the purpose of this paper to examine this method, to develop the correlating parameters, and to test the approach with the data.

SIMILARITY ANALYSIS

In this section the parameters for two-phase flow corresponding to the Euler and Reynolds numbers for single-phase flow will be developed. If two flow systems in single-phase flow are dynamically similar, it can be shown that the Reynolds number and the Euler number for the two systems must be equal. (Note that the Euler number is twice the friction factor.) In itself this condition does not provide a relationship between Reynolds number and friction factor. However once the relationship is found from experimental data for one system (the model), the condition of dynamic similarity require the same relationship to apply to all similar systems. The particular grouping of variables which are known as the *Reynolds* and *Euler numbers* emerge naturally from this similarity analysis for single-phase flow. In the work which follows the groupings will be evolved for two-phase flow, and thus the need for the arbitrary definitions used in the past is eliminated. The relationships for single- and two-phase flow will be developed in parallel to clearly demonstrate the analogy of treatment.

Consider two conduits of different size (see Figure 1) containing in the general case two different fluid systems. Points m_A and m_B are two typical points located at geometrically similar positions relative to the boundaries. The flows are assumed to be dynamically similar. The rules of dynamic similarity require that at all corresponding points,

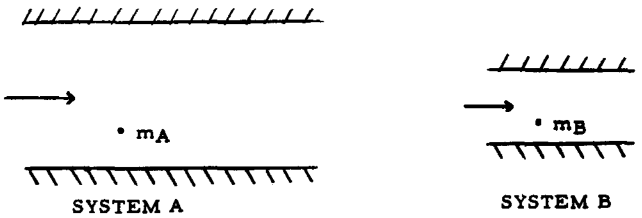

Fig. 1. Sketch of similar systems and two corresponding points.

such as m_A and m_B, the forces and the velocities measured in their own scales are equal. If subscript S is used to indicate the scale quantities, and subscripts A and B refer to the two systems, this statement means that for every set of homologous points

$$\frac{(\text{Inertial force})_A}{F_{SA}} = \frac{(\text{Inertial force})_B}{F_{SB}} \qquad (1a)$$

$$\frac{(\text{Pressure force})_A}{F_{SA}} = \frac{(\text{Pressure force})_B}{F_{SB}} \qquad (1b)$$

$$\frac{(\text{Viscous forces})_A}{F_{SA}} = \frac{(\text{Viscous forces})_B}{F_{SB}} \qquad (1c)$$

$$\frac{V_A}{V_{SA}} = \frac{V_B}{V_{SB}} \qquad (1d)$$

It should be emphasized again that for dynamic similarity to exist the above relationships must apply to each set of corresponding points. They are local relations and the forces and velocities are local velocities, not those which exist over the cross-sectional area of flow as a whole. The force scale at a point is the same for all three types of force. This makes it possible to algebraically rearrange the force equations to give the following relations:

$$\frac{(\text{Inertial force})_A}{(\text{Viscous force})_A} = \frac{(\text{Inertial force})_B}{(\text{Viscous force})_B} \qquad (2a)$$

$$\frac{(\text{Pressure force})_A}{(\text{Inertial force})_A} = \frac{(\text{Pressure force})_B}{(\text{Inertial force})_B} \qquad (2b)$$

The equation for these forces are now developed as they act on a differential volume dv in the vicinity of the point m.

Inertial forces:

Single-phase flow:

$$\frac{\rho}{g_c}\left(V\frac{\partial V}{\partial z} + \frac{\partial V}{\partial t}\right) dv \qquad (3a)$$

Two-phase flow:

$$\left[\frac{\rho_L R_L}{g_c}\left(V_L\frac{\partial V_L}{\partial z} + \frac{\partial V_L}{\partial t}\right) + \frac{\rho_G R_G}{g_c}\left(V_G\frac{\partial V_G}{\partial z} + \frac{\partial V_G}{\partial t}\right)\right] dv \qquad (3b)$$

Viscous forces:

Single-phase flow:

$$\frac{\mu}{g_c}\frac{\partial^2 V}{\partial n^2} dv \qquad (4a)$$

Two-phase flow:

$$\left[\frac{\mu_L}{g_c}\frac{\partial^2 V_L}{\partial n^2} R_L + \frac{\mu_G}{g_c}\frac{\partial^2 V_G}{\partial n^2} R_G\right] dv \qquad (4b)$$

Pressure forces:

Single-phase flow:

$$\frac{\partial P}{\partial z} dv \qquad (5a)$$

Two-phase flow:

$$\frac{\partial P}{\partial z} dv \qquad (5b)$$

Equations (3b), (4b), and (5b) are rigorously correct regardless of the manner in which the two phases happen to be distributed in the differential element. Substituting these quantities into the force ratios of Equation (2) and setting the time derivatives to zero for steady state flow one obtains the following:

Ratio of Inertial to viscous forces:

Single-phase flow:

$$\left\{\frac{\rho V\dfrac{\partial V}{\partial z}}{\mu\dfrac{\partial^2 V}{\partial n^2}}\right\}_A = \left\{\frac{\rho V\dfrac{\partial V}{\partial z}}{\mu\dfrac{\partial^2 V}{\partial n^2}}\right\}_B \qquad (6a)$$

Two-phase flow:

$$\left\{\frac{\rho_L R_L V_L\dfrac{\partial V_L}{\partial z} + \rho_G R_G V_G\dfrac{\partial V_G}{\partial z}}{\mu_L R_L\dfrac{\partial^2 V_L}{\partial n^2} + \mu_G R_G\dfrac{\partial^2 V_G}{\partial n^2}}\right\}_A = \left\{\frac{\rho_L R_L V_L\dfrac{\partial V_L}{\partial z} + \rho_G R_G V_G\dfrac{\partial V_G}{\partial z}}{\mu_L R_L\dfrac{\partial^2 V_L}{\partial n^2} + \mu_G R_G\dfrac{\partial^2 V_G}{\partial n^2}}\right\}_B \qquad (6b)$$

Ratio of pressure to inertial forces:

Single-phase flow:

$$\left\{\frac{\partial P/\partial z}{\dfrac{\rho V}{g_c}\dfrac{\partial V}{\partial z}}\right\}_A = \left\{\frac{\partial P/\partial z}{\dfrac{\rho V}{g_c}\dfrac{\partial V}{\partial z}}\right\}_B \qquad (7a)$$

Two-phase flow:

$$\left\{\frac{\partial P/\partial Z}{\rho_L R_L V_L\dfrac{\partial V_L}{\partial z} + \rho_G R_G V_G\dfrac{\partial V_G}{\partial z}}\right\}_A = \left\{\frac{\partial P/\partial z}{\rho_L R_L V_L\dfrac{\partial V_L}{\partial z} + \rho_G R_G V_G\dfrac{\partial V_G}{\partial z}}\right\}_B \qquad (7b)$$

Equations (6) and (7) result from the conditions of kinetic and geometric similarity. It is now necessary to relate the local quantities to space average ones. For this purpose use is made of the requirement of kinematic similarity (1d) along with the assumption that the scale quantities are independent of position. The resulting transformations for single- and two-phase flow are shown in Figure 2. The transformations for single-phase flow have been developed in a particularly lucid manner by Shames (21). Those presented here for two-phase flow incorporate the additional assumption that kinematic similarity applies to the individual phase velocities as it does to the mixture velocities. The local liquid and gas volume fractions can be related to the average values by the requirement of geometric similarity. That is, at corresponding points, the ratio of the local to average volume fraction in system A must be equal to that in system B.

Single Phase Flow

$$\frac{V_A}{V_B} = \frac{\bar{V}_A}{\bar{V}_B} \qquad \frac{(\partial V/\partial z)_A}{(\partial V/\partial z)_B} = \frac{\bar{V}_A\, l_B}{\bar{V}_B\, l_A}$$

$$\frac{(\partial^2 V/\partial n^2)_A}{(\partial^2 V/\partial n^2)_B} = \frac{\bar{V}_A\, l_B^2}{\bar{V}_B\, l_A^2}$$

Two Phase Flow

$$\frac{V_{LA}}{V_{LB}} = \frac{\bar{V}_{LA}}{\bar{V}_{LB}} \qquad \frac{V_{GA}}{V_{GB}} = \frac{\bar{V}_{GA}}{\bar{V}_{GB}}$$

$$\frac{(\partial V_L/\partial z)_A}{(\partial V_L/\partial z)_B} = \frac{\bar{V}_{LA}\, l_B}{\bar{V}_{LB}\, l_A} \qquad \frac{(\partial V_G/\partial z)_A}{(\partial V_G/\partial z)_B} = \frac{\bar{V}_{GA}\, l_B}{\bar{V}_{GB}\, l_A}$$

$$\frac{(\partial^2 V_L/\partial n^2)_A}{(\partial^2 V_L/\partial n^2)_B} = \frac{\bar{V}_{LA}\, l_B^2}{\bar{V}_{LB}\, l_A^2} \qquad \frac{(\partial^2 V_G/\partial n^2)_A}{(\partial^2 V_G/\partial n^2)_B} = \frac{\bar{V}_{GA}\, l_B^2}{\bar{V}_{GB}\, l_A^2}$$

Fig. 2. Relation between local and average quantities as dictated by conditions of kinematic similarity.

For single-phase flow the transformations of Figure 2 can be used to solve for the local velocity of one of the systems, say system A, and for its gradients in terms of the local velocity of the other system and the average values. For example the local velocity V and its z derivative are

$$V_A = V_B \frac{\bar{V}_A}{\bar{V}_B};$$

$$(\partial V/\partial z)_A = (\partial V/\partial z)_B \frac{\bar{V}_A}{\bar{V}_B}\frac{l_B}{l_A}$$

When these expressions are substituted into Equations (6*a*) and (7*a*) the local quantities are seen to cancel and the groups which emerge are the familiar Reynolds and Euler numbers:

$$\left[\frac{l\bar{V}\rho}{\mu}\right]_A = \left[\frac{l\bar{V}\rho}{\mu}\right]_B = N_{\mathrm{Re}} \tag{8a}$$

$$\left\{\frac{\partial P/\partial z}{\dfrac{\rho \bar{V}^2}{g_c l}}\right\}_A = \left\{\frac{\partial P/\partial z}{\dfrac{\rho \bar{V}^2}{g_c l}}\right\}_B = N_{\mathrm{Eu}} = 2f \tag{8b}$$

The same procedure can be followed for two-phase flow and the analogous expressions developed. As this procedure is followed, it soon becomes clear that for two-phase flow all of the local quantities do not cancel as they conveniently did for single-phase flow. The groups which result after some rearrangement are:

$$N_{\mathrm{ReTP}} = l\bar{V}_m \left[\frac{\rho_L \bar{R}_L (\bar{V}_L/\bar{V}_m)^2 + \rho_G \bar{R}_G (\bar{V}_G/\bar{V}_m)^2 C_1}{\mu_L \bar{R}_L (\bar{V}_L/\bar{V}_m) + \mu_G \bar{R}_G (\bar{V}_G/\bar{V}_m) C_2}\right] \tag{9a}$$

$$N_{\mathrm{EuTP}} = 2f = \left\{\frac{\partial P/\partial z}{\dfrac{\bar{V}_m^2}{g_c l}}\right\}$$

where

$$\left\{\frac{1}{\rho_L \bar{R}_L (\bar{V}_L/\bar{V}_m)^2 + \rho_G \bar{R}_G (\bar{V}_G/\bar{V}_m)^2 C_1}\right\} \tag{9b}$$

$$\bar{V}_m = \frac{Q_L + Q_G}{A} \tag{10}$$

$$C_1 = \frac{R_G}{\bar{R}_G}\frac{\bar{R}_L}{R_L}\left(\frac{\bar{V}_L}{\bar{V}_G}\right)^2 \frac{V_G}{V_L}\frac{dV_G/dz}{dV_L/dz} \tag{11a}$$

$$C_2 = \frac{R_G}{\bar{R}_G}\frac{\bar{R}_L}{R_L}\frac{\bar{V}_L}{\bar{V}_G}\frac{d^2V_G/dn^2}{d^2V_L/dn^2} \tag{11b}$$

The ratio of the phase velocities to the mixture velocity can be expressed in terms of the input volume fraction liquid λ and the in-place volume fractions R_L and R_G:

$$\frac{\bar{V}_L}{\bar{V}_m} = \frac{\lambda}{\bar{R}_L} \tag{12a}$$

$$\frac{\bar{V}_G}{\bar{V}_m} = \frac{1-\lambda}{\bar{R}_G} \tag{12b}$$

$$\lambda = \frac{Q_L}{Q_L + Q_G} \tag{12c}$$

Thus Equations (9*a*) and (9*b*) can be written as follows:

$$N_{\mathrm{ReTP}} = l\bar{V}_m \left\{\frac{\rho_L \dfrac{\lambda^2}{\bar{R}_L} + \rho_G \dfrac{(1-\lambda)^2}{\bar{R}_G} C_1}{\mu_L \lambda + \mu_G (1-\lambda) C_2}\right\} \tag{13a}$$

$$N_{\mathrm{EuTP}} = 2f = \left\{\frac{\partial P/\partial z}{\dfrac{\bar{V}_m^2}{g_c\, l}}\right\}\left\{\frac{1}{\rho_L \dfrac{\lambda^2}{\bar{R}_L} + \rho_G \dfrac{(1-\lambda)^2}{\bar{R}_G} C_2}\right\} \tag{13b}$$

Thus the correct expressions for the Reynolds and Euler numbers clearly emerge from this type of analysis. Before it is possible to write completely definitive equations, some decision will have to be made about the C terms. However it is of interest that the definition for mixture density and mixture viscosity are no longer arbitrary. If dynamic similarity is to exist, the mixture properties are defined by

$$\rho_{\mathrm{TP}} = \rho_L \frac{\lambda^2}{\bar{R}_L} + \rho_G \frac{(1-\lambda)^2}{\bar{R}_G} C_1 \tag{14a}$$

$$\mu_{\mathrm{TP}} = \mu_L \lambda + \mu_G (1-\lambda) C_2 \tag{14b}$$

The problem now comes down to finding values for the C terms.

SOME SPECIAL CASES

As in most problems of this type it is necessary to exercise intuitive judgment to complete the analysis. As a basis for arriving at this judgment the equations resulting from several assumptions for the values of the C terms are presented.

Case I: No Slip and Homogeneous Flow

Under these conditions

$$C_1 = C_2 = 1.0 \tag{15a}$$

$$\lambda = R_L \tag{15b}$$

$$1 - \lambda = R_G \tag{15c}$$

Furthermore since these equations must produce the single-phase groupings as the amount of either phase goes to zero, the characteristic length l for tube flow must be the diameter D. This produces the following results:

$$N_{\mathrm{ReNS}} = \frac{4W_T}{\pi D \mu_{\mathrm{NS}}} \tag{16a}$$

$$f_{\mathrm{NS}} = \frac{\partial P/\partial z}{\dfrac{2\, G_T^2}{g_c \rho_{\mathrm{NS}} D}} \tag{16b}$$

The mixture properties now can be obtained from Equations (14a) and (14b):

$$\rho_{NS} = \rho_L \lambda + \rho_G (1-\lambda) \qquad (17a)$$

$$\mu_{NS} = \mu_L \lambda + \mu_G (1-\lambda) \qquad (17b)$$

Numerous arbitrary definitions of the viscosity and the density for homogeneous flow have been proposed in the past. For homogeneous flow Bankoff (12) has suggested weighting the gas and liquid viscosities by the volume fraction gas and liquid R_G and R_L:

$$\mu_{NS} = R_G \mu_G + R_L \mu_L \qquad (18a)$$

The CISE group (6) has proposed the equation

$$\mu_{NS} = x \mu_G + (1-x) \mu_L \qquad (18b)$$

Woods et al. (13) have suggested

$$\frac{1}{\mu_{NS}} = \frac{x}{\mu_G} + \frac{1-x}{\mu_L} \qquad (18c)$$

In these equations x is the weight fraction vapor. Similarly it has been suggested (8) that the density be weighted by the in-place fraction liquid and gas:

$$\rho_{NS} = R_L \rho_L + R_G \rho_G \qquad (18d)$$

It is now quite clear that the definitions for mixture properties are not arbitrary. For homogeneous, no slip flow the proper definitions are given by Equations (17a) and (17b) with the correct weighting factor being the flowing volume fraction of liquid.

The general equations for the Reynolds number and the friction factor can now be written in terms of the no slip values:

$$N_{ReTP} = N_{ReNS} \frac{(\rho_L/\rho_{NS}) \frac{\lambda^2}{\bar{R}_L} + (\rho_G/\rho_{NS}) \frac{(1-\lambda)^2}{\bar{R}_G} C_1}{(\mu_L/\mu_{NS}) \lambda + (\mu_G/\mu_{NS})(1-\lambda) C_2} \qquad (19a)$$

$$f = f_{NS} \frac{1}{(\rho_L/\rho_{NS}) \frac{\lambda^2}{\bar{R}_L} + (\rho_G/\rho_{NS}) \frac{(1-\lambda)^2}{\bar{R}_G} C_1} \qquad (19b)$$

Case II: Slip Takes Place but C_1 and C_2 are Assumed to be 1.0:

If it is assumed that the ratio of each phase velocity to the average velocity remains constant over the cross section, as was done by Levy (20), but if this ratio is permitted to be different from 1.0, then

$$N_{ReTP} = N_{ReNS} \left[(\rho_L/\rho_{NS}) \frac{\lambda^2}{\bar{R}_L} + (\rho_G/\rho_{NS}) \frac{(1-\lambda)^2}{\bar{R}_G} \right] \qquad (20a)$$

$$f_{TP} = f_{NS} \frac{1}{\left[(\rho_L/\rho_{NS}) \frac{\lambda^2}{\bar{R}_L} + (\rho_G/\rho_{NS}) \frac{(1-\lambda)^2}{\bar{R}_G} \right]} \qquad (20b)$$

Here both the flowing volume fraction and the in-place volume fraction of each phase enter into the definition of the mixture density.

Isbin et al. (22) have reviewed the various equations proposed for friction factor and Reynolds number which include the fraction liquid and vapor holdup. Their review demonstrates the totally empirical nature of the equations. Now it can be seen that the definition for f and Re depend on the assumptions made about the C terms.

Case III: C_1 and C_2 are Assumed to be Zero:

Under conditions where R_G is small, a case can be made for assuming C_1 and C_2 to be near zero. Both the Reynolds number and friction factor then are defined in terms of liquid properties:

$$N_{ReTP} = \frac{4 W_T}{\pi D \mu_L} \frac{1-x}{\bar{R}_L} \qquad (21a)$$

$$f_{TP} = \frac{\partial P/\partial z}{\frac{2 G_T^2}{\rho_L g_c D} \frac{(1-x)^2}{\bar{R}_L}} \qquad (21b)$$

The similarity between these equations and the many suggested correlations of two-phase flow based on liquid flow rates and properties is apparent. The form suggested by Flinn and Lottes (23) reduces to

$$N_{ReTP} = \frac{4 W_T}{\pi D \mu_L} \qquad (22a)$$

$$f_{TP} = \frac{\partial P/\partial z}{\frac{2 G_T^2}{\rho_L g_c D} \frac{1}{\bar{R}_L^2}} \qquad (22b)$$

The modification of Petrick (24) can be written as

$$N_{ReTP} = \frac{4 W_T}{\pi D \mu_L} \qquad (23a)$$

$$f_{TP} = \frac{dP/dz}{\frac{2 G_T^2}{\rho_L g_c D} \frac{(1-x)^2}{R_L^2}} \qquad (23b)$$

The form suggested by Chisholm and Laird (25) is

$$N_{ReTP} = \frac{4 W_L}{\pi D \mu_L} \qquad (24a)$$

$$f_{TP} = \frac{dP/dz}{\frac{2 G_T^2}{\rho_L g_c D} \frac{0.8}{R_L^{1.75}}} \qquad (24b)$$

Other similar types of formulas have been noted by Isbin (22). Note that the modification to the friction factor equation which was investigated by Petrick is closest to this case and could be expected to be correct for situations of relatively high values of R_L. However none of these methods appear to recognize that modifications to the definitions of the friction factor must be accompanied by modifications to the definition of the Reynolds number as well in order to meet requirements of similarity.

Case IV: $C_1 = C_2 = \bar{V}_L/\bar{V}_G$

If one assumes that the ratio of the gas velocity gradient and the liquid velocity gradient are constant in the flow direction and in the direction normal to flow, and that the velocity ratios are constant over the cross section, as was done in Case II, then the relations for C_1 and C_2 reduce to

$$C_1 = C_2 = \frac{\bar{V}_L}{\bar{V}_G} \qquad (25)$$

When the proper substitutions are made in Equations (9a) and (9b), the equations which result are

$$N_{ReTP} = \frac{4 W_T}{\pi D} \frac{1}{\mu_L R_L + \mu_G R_G} \qquad (26a)$$

$$f_{TP} = \frac{\partial P/\partial z}{\frac{2 G_T^2}{\rho_{NS} g_c D} \left(\frac{\lambda}{R_L} \right)} \qquad (26b)$$

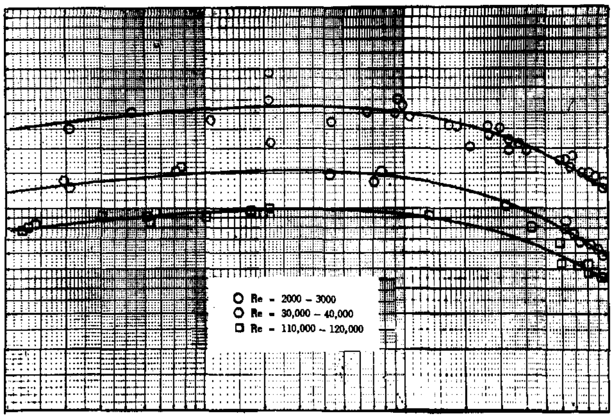

Fig. 3. Typical Case II correlating curves.

It is interesting to note that the definition for the Reynolds number which emerges from this series of assumptions is identical to that proposed by Hughmark (26). However once again the definition which he used for the friction factor is not consistent with the assumptions which lead to the Reynolds number.

It is now quite clear that by properly selecting values for the two C terms it is possible to develop either the friction factor or the Reynolds number expression used by many (if not all) of the previous investigators. It is likewise evident that these earlier approaches were not consistent within themselves.

this correlation was

$$f = 0.00140 + \frac{0.125}{(Re)^{0.32}} \tag{27}$$

Case II

This correlation was constructed by using experimental data to establish the graphical relationship between f and Re, as defined by Equations (20*a*) and (20*b*). In this instance it was necessary to have data which provided not only the measured frictional pressure drop and the operating variables but experimentally determined values of holdup as well. As discussed in Part A certain of the culled data included information on fraction liquid holdup. Of a total of approximately 2,400 data points holdup data were reported for about 800 points. Data taken in small diameter lines were not usable because the pressure drops obtained included large acceleration effects. Data which reported holdup values less than 0.10 fraction volume of the tube were eliminated as being of doubtful accuracy. In all approximately 400 data points of the total 2,400 culled data points were used to construct the correlation.

For each of these 400 experimental data points the friction factor and Reynolds number were calculated from Equations (20*a*) and (20*b*). The results, when plotted, revealed a distinct trend with flowing fraction volume liquid. Some typical results are shown in Figure 3. When these various curves were normalized with single-phase friction factor calculated for the mixture Reynolds number, the various curves for narrow ranges of Reynolds numbers converged into essentially one curve (Figure 4). The equation for the normalized curve of Figure 4 is

$$\alpha(\lambda) = \frac{f}{f_o} = 1.0 + \frac{-\ln\lambda}{1.281 - 0.478(-\ln\lambda) + 0.444(-\ln\lambda)^2 - 0.094(-\ln\lambda)^3 + 0.00843(-\ln\lambda)^4} \tag{28}$$

The problem now becomes that of determining which of these assumptions leads to the most satisfactory correlation.

CORRELATION FOR CASES I AND II

Once the methods of similarity have been used to correctly define the parameters, experimental data are utilized to develop a relationship between these parameters, thus constructing the desired correlation. The fact that similarity analysis must rest finally on experimental data is not particularly disturbing. All existing correlations for turbulent flow, even those which have been evolved for single-phase flow with the basic equations of motion used, depend on experiment to provide certain of the correlating constants. The similarity method does this by utilizing some of the data (the model) to develop a relationship which will apply equally well to other data taken at other conditions and in other systems.

This section presents the correlations developed for Cases I and II. The correlations were then tested with the culled experimental data described in Part A. Results of these tests are presented in the next section.

Case I

The assumptions of homogeneous flow which underlie this case permit the two-phase mixture to be treated as an equivalent single fluid with the properties defined by Equations (17*a*) and (17*b*). Data for single-phase flow have already provided the relation between the friction factor and Reynolds number for dynamically similar single-phase systems. For dynamically similar homogeneous two-phase systems the same *f-Re* relation must apply, providing f and Re are defined by Equations (16*a*) and (16*b*). The equation relating single-phase *f-Re* used for

The method for calculating two-phase pressure drop by this correlation can now be outlined. The Reynolds number for the mixture is determined from

$$N_{ReTP} = \frac{4\,W_T}{\pi\,D\mu_{NS}}\,\beta \tag{29}$$

$$\beta = \frac{\rho_L}{\rho_{NS}}\frac{\lambda^2}{\bar{R}_L} + \frac{\rho_G}{\rho_{NS}}\frac{(1-\lambda)^2}{\bar{R}_G} \tag{30}$$

ρ_{NS} and μ_{NS} are defined by Equations (17*a*) and (17*b*) for the case of no slip. The frictional pressure drop is determined from the following equation, obtained by combining Equations (20*b*) and (28):

$$\frac{\partial P}{\partial z} = \frac{2\,G_T^2\,f_o}{g_c\,D\rho_{NS}}\,\alpha(\lambda)\beta \tag{31}$$

$\alpha(\lambda)$ is given by Equation (28). This equation can be rearranged as follows:

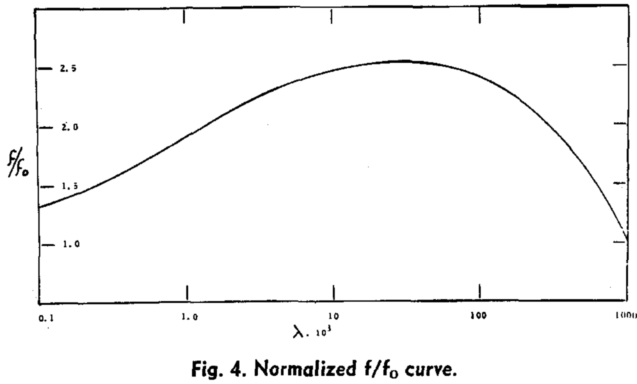

Fig. 4. Normalized f/f_0 curve.

$$\frac{\partial P}{\partial z} = \left[\frac{2\, G_T^2 f_o}{g_c\, D\, \rho_L}\right] \frac{\rho_L}{\rho_{NS}} \alpha(\lambda)\beta \qquad (32a)$$

$$\frac{\partial P'}{\partial z} = \left[\frac{2\, G_T^2 f_o}{g_c\, D\, \rho_G}\right] \frac{\rho_G}{\rho_o} \alpha(\lambda)\beta \qquad (32b)$$

The terms in brackets in each of these equations represent a single-phase frictional pressure drop. In Equations (32*a*) this group represents the frictional pressure drop which would exist if the mixture flows as a liquid, the friction factor f_o being calculated from Equation (27) with the mixture Reynolds number as defined by Equation (29) used. Similarly in Equation (32*b*), the bracketed term is calculated on the assumption that the mixture flow with the density of the gas, but with the friction factor evaluated at the mixture Reynolds number. These equations can be rearranged to form the familiar parameters of Martinelli:

$$\phi^2_{LM} = \frac{\rho_L}{\rho_{NS}} \alpha(\lambda)\beta \qquad (33a)$$

$$\phi^2_{GM} = \frac{\rho_G}{\rho_{NS}} \alpha(\lambda)\beta \qquad (33b)$$

The discussion which appeared earlier in this paper showed that two types of ϕ_L terms have been used for correlation. One group correlated in terms of

$$\phi_L = \frac{\partial P/\partial z}{(\partial P/\partial z)_L}$$

where the denominator is the pressure drop calculated as if the liquid phase flowed alone in the tube. Others have used

$$\phi_{Lo} = \frac{dP/dz}{(dP/dz)_{Lo}}$$

Here the denominator is the frictional pressure drop when the total mass flows as a liquid. The Reynolds number, based on the total mass flowing as a liquid, is used to calculate the friction factor.

It is now clear that both of these methods are incorrect.

EVALUATION OF CORRELATIONS FOR CASES I AND II

Each of these correlations were evaluated by comparing the calculated and measured pressure drop for each of the culled data points. Calculated pressure drops were obtained by adding the pressure drop due to friction (calculated from the correlation) to that due to acceleration of the gas and liquid along the tube.

For the homogeneous flow of Case I the equation for calculated pressure drop is

$$\left(\frac{\partial P}{\partial z}\right)_c = \left(\frac{\partial P}{\partial z}\right) / 1 - Acc \qquad (34a)$$

$$Acc = \frac{16\, W_T W_G\, \bar{P}}{\pi^2\, g_c\, D^4\, P_1 P_2\, \bar{\rho}_G} \qquad (34b)$$

$\left(\frac{\partial P}{\partial z}\right)_c$ is the frictional pressure drop as calculated from Equations (16*a*) and (16*b*). P_1 and P_2 represent the absolute static pressures at the upstream and downstream positions, while $\bar{P}$ and $\bar{\rho}_G$ are the arithmetic mean static pressure and the gas density respectively. Note that while the determination of frictional pressure drop is explicit, the calculation of the total pressure drop involves an iterative procedure.

For Case II the calculated pressure drop is determined from

$$\left(\frac{\partial P}{\partial z}\right)_c = \left(\frac{\partial P}{\partial z}\right) + \frac{1}{g_c\, A^2\, \Delta z} \left[W_G^2\, \Delta\left(\frac{1}{\rho_G \bar{R}_G}\right) + \frac{W_L^2}{\rho_L} \Delta\left(\frac{1}{\bar{R}_L}\right)\right] \qquad (35)$$

The frictional pressure drop is determined from Equation (31). Once again an iterative procedure is required in order to evaluate the delta terms. Use of this correlation requires a method for the prediction of holdup. Several holdup correlations were evaluated in the first paper. Although none of these proved to be fully satisfactory over the entire range of data, the method suggested by

TABLE 1. TEST OF PRESSURE DROP CORRELATIONS, TWO-COMPONENT SYSTEMS, DISTRIBUTION BY LINE SIZE LIQUID VISCOSITY

NOMINAL TUBE SIZE	NOMINAL LIQUID VISCOSITY	LOCKHART – MARTINELLI			CASE I			CASE II			NO. DATA POINTS
Inches	C_p	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	j
1	1	−6.6	10.1	10.0*	− 9.4	17.9	17.5	−25.2	18.2	13.5	224
	3	3.8	29.1	20.0	− 1.0	29.7	20.0	8.6	24.8	12.0*	230
	20	−5.5	24.7	20.0	0.0	46.7	25.0	6.7	24.4	18.6*	156
2	1	9.2	37.7	25.0	−11.2	13.8	12.5	2.4	18.4	15.5*	320
	3	−4.7	22.9	25.0	− 1.4	39.4	25.0	1.6	19.7	16.0*	398
	20	13.2	52.0	30.0	−19.1	22.6	22.5	10.3	27.2	20.0*	401
3½	1	31.0	50.2	47.5	−18.5	29.2	22.5	− 0.3	26.8	26.2*	109
	3	16.3	39.3	22.5	− 0.9	34.6	25.0*	9.3	24.9	25.0	67
	20	−0.4	26.2	22.5*	7.3	33.5	25.0	10.6	24.5	18.6	111
5	1	38.3	12.2	12.5	16.4	20.0	17.5*	50.6	18.8	19.3	24
	3	11.6	41.5	37.5	2.0	31.7	25.0	11.2	19.2	16.0*	131
	20	−1.0	25.0	25.0	7.2	29.8	20.0	7.3	22.7	14.0*	122

Hughmark (*10*) appeared to give the best overall agreement and was used to determine the holdup for use in Equation (30).

Equations (34*a*) and (35) along with the corresponding expressions for the frictional pressure drop were programed for the electronic computer. Calculated pressure drops were then obtained for each of the approximately 2,400 culled data points as predicted by the correlations of Case I and Case II. Deviations between calculated and measured values were calculated for each point, the data grouped, and the results processed through an error distribution program to generate a histogram of the deviations and the significant statistical parameters:

$$d = \frac{\left(\frac{\partial P}{\partial z}\right)_c - \left(\frac{\partial P}{\partial z}\right)_m}{(\partial P/\partial z)_m} \quad (36)$$

$$\bar{d} = \frac{\Sigma d}{j} \quad (37)$$

$$\sigma = \left\{ \frac{\Sigma d^2 - \frac{(\Sigma d)^2}{j}}{j - 1} \right\}^{1/2} \quad (38)$$

In addition the histogram for each group was counted to obtain the quantity ψ, which defines width of the band around the mean which includes approximately 68% of the data points in the group. Results of these calculations appear in Tables 1 to 3.

In Table 1 all the two component data are grouped by line size and the viscosity of the liquid phase. Included also are the statistical parameters for the Martinelli correlation, shown in the first part of this paper to be the best of those tested. In general the Case I correlation is superior to the Martinelli correlation, and the Case II correlation is more accurate than Case I. For each data group an asterisk has been placed under the column of the correlation which gives the best agreement between calculated and measured values. Of the twelve groups Case II correlation gives best agreement for eight groups, the Case I correlation for two, and the Martinelli correlation for two. The Martinelli correlation performed best for low viscosity liquids in small tube sizes, conditions which produced most of the data used by Lockhart and Martinelli to construct their correlation. In the second instance where the Martinelli correlation performed best, the Case II correlation was almost as satisfactory.

The standard deviation σ is consistently lower when the Case II correlation is used than when Case I is used. Case I consistently produces lower values of σ than Martinelli. This quantity, which is a measure of the scatter about the mean, exceeds 35% for five groups with the Martinelli correlation, two groups for Case I, and no groups for Case II. Martinelli produces values of σ exceeding 25% for eight of the twelve groups. Case II correlation produces only two σ's exceeding 25%.

TABLE 2. TEST OF PRESSURE DROP CORRELATIONS, TWO-COMPONENT SYSTEMS, DISTRIBUTION OF FLOW REGIME

FLOW TYPE	MARTINELLI			CASE I			CASE II			NO. DATA POINTS
	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	j
Plug	9.4	36.3	20.0	−12.3	12.7	10.0*	9.5	18.0	15.1	270
Stratified	23.3	33.0	22.5	− 0.3	30.8	25.0*	13.4	30.3	17.5	34
Wave	38.4	85.7	42.5	9.0	33.5	30.0	11.5	22.6	20.5*	287
Slug	2.9	31.2	17.5	− 2.9	29.4	15.0*	9.5	21.7	15.6	974
Annular	−12.8	35.6	30.0	−21.7	46.3	37.5	−11.2	26.0	19.0*	267
Dispersed	18.0	34.1	25.0	3.1	31.1	27.5	14.8	16.9	17.1*	111

A comparison of correlations on data grouped by observed flow regime appear in Table 2. Each of the flow regime groups includes data at all line sizes and for varying fluid properties. It is seen that the Case I and Case II correlations are in all instances better than the Martinelli correlation. As indicated by the asterisks, agreement is better for three groups using Case I and for three groups using Case II correlation.

Results for the one-component, steam-water data in Table 3 show that the Martinelli correlation is most satisfactory for the low- and the high-pressure data, but Case I is significantly better at the intermediate pressure level. For all three groups the Case II correlation is poor. This poor agreement results from the fact that the Hughmark correlation describes the experimental holdup for these steam water tests with rather large deviations. These deviations reflect in the prediction of the pressure drop values.

It is of some interest to reflect on Case I. This correlation, based on the assumptions of homogenous flow with the definition of the mixture properties dictated by requirements of similarity, was constructed without the use of any two-phase flow pressure drop data. Despite this fact it predicts the pressure drop with greater reliability than does the Martinelli correlation for ten of the twelve two-component groups and in many cases is almost as good as Case II. For the one-component, small tube-size data it is likely that the larger errors are due primarily to inaccuracies in the prediction of the acceleration component.

SUMMARY AND CONCLUSIONS

Various correlating parameters for frictional pressure drop are developed, starting with the requirement for dynamic similarity for two-phase flow. Two cases are explored in detail and tested with the culled data. Statistical terms describing the deviation between the prediction of the correlations and the measured data are compared. In

TABLE 3. TEST OF PRESSURE DROP CORRELATIONS, ONE-COMPONENT, STEAM-WATER SYSTEM

PRESSURE RANGE	TUBE DIAMETER	MARTINELLI			CASE I			CASE II			NO. DATA POINTS
PSIA	Inches	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	$\bar{d}$	σ	ψ	j
25 – 100	1.06	39.5	66.6	55.0*	61.5	179	95.0	132	153	–	93
400 – 800	0.484	63.6	57.0	52.5	−10.2	44.6	37.5*	71.0	70.0	–	103
1000 – 1400	0.484	−17.5	21.1	17.5*	−29.4	16.2	10.0	32.6	29.6	25.1	131

general the Case II correlation based on holdup gives better agreement with data than does earlier correlations. As methods for predicting holdup are improved, these correlations should be even more satisfactory.

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NOTATION

A = conduit cross-sectional area
C_1, C_2 = grouping of variables defined by Equations (11*a*) and (11*b*)
d = percentage deviation between calculated and measured pressure gradient
$\bar{d}$ = mean deviation for j data points
D = tube diameter
F_S = local scale of force
f = friction factor, single-phase flow
f_{NS} = friction factor, two-phase flow based on conditions of no slip and homogeneous flow, Equation (16*b*)
f_o = friction factor for single phase flow evaluated at the mixture Reynolds number, Equation (30)
f_{TP} = friction factor, two-phase flow
g_c = dimensional constant of Newton's equation of motion
G_T = mass velocity based on total flow rate of liquid plus gas
j = number of data points in a test group
l = local scale for length
n = normal to the flow direction
N_{Eu} = Euler number
N_{Re} = Reynolds number, single-phase flow
N_{ReNS} = Reynolds number, two-phase flow under conditions of no slip, Equation (16*a*)
N_{ReTP} = Reynolds number, two-phase flow
n = distance measured normal to the flow direction
$(\partial P/\partial z)$ = pressure gradient due to friction
$(\partial P/\partial z)_c$ = total pressure gradient due to friction and acceleration
P = local pressure
$\bar{P}$ = average pressure
Q_L, Q_G = volumetric flow rate of liquid, gas
R_L, R_G = local volume faction liquid or gas in place
$\bar{R}_L$, $\bar{R}_G$ = average, in place volume faction liquid, gas
t = time
v = volume
V = local velocity, single-phase flow
$\bar{V}$ = average velocity, single-phase flow
V_L, V_G = local velocity, liquid, gas
$\bar{V}_L$, $\bar{V}_G$ = average velocity of liquid, gas
$\bar{V}_m$ = average velocity of gas-liquid mixture, defined by Equation (10)
V_s = local scale of velocity
W_T = total mass flow rate of liquid and gas
W_L, W_G = mass flow rate of liquid, gas
x = ratio of weight of vapor flowing
z = direction of flow

Greek Letters

β = dimensionless group defined by Equation (30)
λ = ratio of the volumetric flow rate of liquid to the total volumetric flow rate, Equation (12*c*)
μ = viscosity, single-phase
μ_L, μ_G = viscosity, liquid, gas
μ_{NS} = viscosity of two-phase homogeneous mixture, defined by Equation (17*b*)
μ_{TP} = viscosity of two-phase mixture
ρ = density, single phase
ρ_L, ρ_G = density of liquid, gas
ρ_{NS} = density of two-phase homogeneous mixture, defined by Equation (17*a*)
ρ_{TP} = density of two-phase mixture
σ = standard deviation
ϕ_L = ratio of the two-phase pressure gradient to the pressure gradient if liquid flowed alone in the conduit
ϕ_{Lo} = ratio of the two-phase pressure gradient to the gradient if both phases flowed as a liquid
ϕ_{LM} = ratio of the two-phase pressure gradient to the gradient if both phases flowed as a liquid at the mixture Reynolds number

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